Published by Institute of Physics Publishing for SISSA

Received: *April 2, 2006*
Accepted: *June 21, 2006*
Published: *July 11, 2006*

String theory dual of the β-deformed gauge theory

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Abstract: We consider the AdS/CFT correspondence between the β-deformed supersymmetric gauge theory and the type IIB string theory on the Lunin-Maldacena background. Guided by gauge theory results, we modify and extend the supergravity solution of Lunin and Maldacena in two ways. First we make it to be doubly periodic in the deformation parameter, $\beta \to \beta + 1$ and $\beta \to \beta + \tau_0$, to match the β-periodicity property of the dual gauge theory. Secondly, we reconcile the $\mathrm{SL}(2, \mathbb{Z})$ symmetry of the gauge theory, which acts on the constant parameters τ_0 and β, with the $\mathrm{SL}(2, \mathbb{Z})$ invariance of the string theory, which involves the dilaton-axion field $\tau(\mathbf{x})$. Our proposed modified configuration transforms correctly under the $\mathrm{SL}(2, \mathbb{Z})$ of string theory when its parameters are transformed under the $\mathrm{SL}(2, \mathbb{Z})$ of the gauge theory. We interpret the resulting configuration as the string theory (rather than supergravity) background which is dual to the β-deformed conformal Yang-Mills. Finally, we check that our string theory background leads to the IIB effective action which is correctly reproduced by instanton calculations on the gauge theory side, carried out at weak coupling, in the large-N limit, but to all orders in the deformation parameter β.

Keywords: AdS-CFT Correspondence, Duality in Gauge Field Theories, Solitons Monopoles and Instantons, Supersymmetry and Duality.

Contents

1. Introduction

The purpose of this paper is to discuss certain general properties of the AdS/CFT correspondence between the exactly marginal β-deformed gauge theories and the type IIB string theory. In its original formulation, the AdS/CFT duality [1] relates the string theory on a curved background $AdS_5 \times S^5$ to the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory living on the boundary of AdS_5. All symmetries are known to match precisely on the left and on the right hand side of the correspondence. In particular, the $\mathrm{SL}(2,\mathbb{Z})$ S-duality of the $\mathcal{N} = 4$ SYM becomes the $\mathrm{SL}(2,\mathbb{Z})$ duality of IIB string theory.

β-deformations of the $\mathcal{N} = 4$ supersymmetric Yang-Mills define a family of conformally-invariant four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories. The AdS/CFT duality extends to the β-deformed theories where it relates the β-deformed $\mathcal{N} = 4$ SYM and the supergravity on the deformed $AdS_5 \times \tilde{S}^5$ background. The gravity dual was found by Lunin and Maldacena in ref. [2], and this provides a foundation for a precise formulation of the AdS/CFT duality in the deformed case. In a recent paper [3] the supergravity dual and the resulting string theory effective action were successfully tested and studied using Yang-Mills instanton methods developed earlier for the $\mathcal{N} = 4$ case in [4−7]. The approach of [3] was based on the semiclassical (i.e. weak coupling) approximation and, in order to compare with the Lunin-Maldacena solution, the results were further restricted to the small-deformation limit $\beta \ll 1$ with $\beta^2 N$ fixed, in the usual large-N limit.

It is known that the β-deformed gauge theory retains the $\mathrm{SL}(2,\mathbb{Z})$ duality of the parent $\mathcal{N} = 4$ SYM [8]. At the same time, the IIB string theory is invariant under the $\mathrm{SL}(2,\mathbb{Z})$ transformations of string theory. In the present paper we want to study the

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AdS/CFT correspondence between the gauge and the string theory for general values of the deformation parameter β, as well as the interplay between the SL$(2,\mathbb{Z})$ duality on the gauge theory and on the string theory side.

Marginal deformations of the SYM have also been studied extensively in [9 − 13]. More recently, several perturbative calculations in the β-deformed theories were carried out in [14 − 16] where it was noted that there are many similarities between the deformed and the undeformed theories which emerge in perturbation theory in the large number of colours limit. In [16] it was shown that for real values of β all perturbative scattering amplitudes in the β-deformed theory are completely determined by the corresponding $\mathcal{N} = 4$ amplitudes. Studies of perturbative integrability of β-deformations were conducted in refs. [17 − 19].

The β-deformed SYM is a conformal $\mathcal{N} = 1$ supersymmetric gauge theory obtained by an exactly marginal deformation of the superpotential of the $\mathcal{N} = 4$ SYM. In terms of the three adjoint chiral superfields of the $\mathcal{N} = 4$ theory, the deformation takes the form:

$$\mathcal{W} = ig\,\mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \;\rightarrow\; ih\,\mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2)\ . \tag{1.1}$$

The deformed superpotential preserves one $\mathcal{N} = 1$ supersymmetry of the original $\mathcal{N} = 4$ SYM and leads to a theory with a global $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry [2]

$$\begin{aligned} \mathrm{U}(1)_1: &\qquad (\Phi_1, \Phi_2, \Phi_3) \rightarrow (\Phi_1, e^{i\varphi_1}\Phi_2, e^{-i\varphi_1}\Phi_3)\,, \\ \mathrm{U}(1)_2: &\qquad (\Phi_1, \Phi_2, \Phi_3) \rightarrow (e^{-i\varphi_2}\Phi_1, e^{i\varphi_2}\Phi_2, \Phi_3)\ . \end{aligned} \tag{1.2}$$

At the classical level the deformation in (1.1) is marginal (the superpotential has classical mass dimension three) and the deformed theory is parameterized by three complex constants, h, β, τ_0, with τ_0 being the usual complexified gauge coupling.

At quantum level, this deformation is not exactly marginal since the operators in (1.1) can develop anomalous dimensions. Using constraints of $\mathcal{N} = 1$ supersymmetry, and the exact NSVZ beta function, Leigh and Strassler [9] argued that the deformation (1.1) is marginal at quantum level subject to a *single* complex constraint on the three parameters, $\gamma(h, \beta, \tau_0) = 0$. Here the function γ is the sum of the anomalous dimensions γ_i of the three fields Φ_i, so that $\gamma = \sum_{i=1,2,3}\gamma_i$. This constraint implies that there is a 2-complex-dimensional surface $\gamma(h, \beta, \tau_0) = 0$ of conformally invariant $\mathcal{N} = 1$ theories[1] obtained by deforming $\mathcal{N} = 4$ SYM. From now on, we will always assume that the Leigh-Strassler conformal constraint is formally resolved in terms of $h = h(\tau_0, \beta)$, and that the parameter h is eliminated. This implies that the β-deformed theory is a conformal $\mathcal{N} = 1$ supersymmetric gauge theory which is characterized by two mutually independent complex constants, β and τ_0.

Furthermore, it has been argued by Dorey, Hollowood and Kumar in [8] that the β-deformed SYM retains the SL$(2,\mathbb{Z})$ duality of the parent $\mathcal{N} = 4$ theory. More precisely, the

[1] For a special case when the deformation parameter β is real, and in the large N limit, the Leigh-Strassler constraint can be solved to all orders in perturbation theory [15, 16] and gives simply $|h|^2 = g^2$. In this paper we will retain general complex deformations for which no simple solution of the constraint is known to all orders.

authors of [8] showed that at least in the massive phase the deformed theory has an action of the SL$(2,\mathbb{Z})$ group which interchanges various massive vacua of the theory. Under this transformation, the deformation parameter β transforms as a modular form [8]:

$$\tau_0 \to \tau_0' = \frac{a\tau_0 + b}{c\tau_0 + d}\,, \qquad \beta \to \beta' = \frac{\beta}{c\tau_0 + d}\,, \qquad ad - bc = 1,\ a,b,c,d \in \mathbb{Z}\ . \tag{1.3}$$

For the gauge coupling τ_0 to transform in the standard fractional-linear way (1.3), it is required [8] to define it in a particular way,

$$\tau_0 = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} - n\frac{iN}{\pi}\log\frac{h}{g}\ . \tag{1.4}$$

which we will always assume.[2]

The action of the SL$(2,\mathbb{Z})$ transformations in the deformed gauge theory must then persists also in the conformal phase. In what follows, we will call the action of SL$(2,\mathbb{Z})$ in eq. (1.3) — the SYM SL$(2,\mathbb{Z})$ duality. The relation of this duality with the SL$(2,\mathbb{Z})$ duality of the corresponding type IIB string theory turns out to be non-trivial for $\beta \neq 0$. The relation and reconciliation between the two SL$(2,\mathbb{Z})$ transformations on the SYM and on the string side is one of the main subjects of this paper.

The β-deformed gauge theory is also known to be doubly periodic in β. First, it is obvious from the deformation of the superpotential in (1.1) that all observables in the theory must be periodic as $\beta \to \beta + 1$. In fact, all known perturbative and multi-instanton calculations [3] in this theory automatically exhibit this periodicity. Secondly, it was argued in [8] that the gauge theory is also periodic under $\beta \to \beta + \tau_0$. Clearly this double periodicity in β must also be manifest in the dual string theory formulation. However, the β-periodicity in string theory is obscured in the supergravity description. One must use the full string theory background rather than the supergravity solution, which is a good approximation to string theory only for small values of β [2]. In particular, the supergravity dual of Lunin-Maldacena does not exhibit the double β periodicity. The reconciliation of the β-periodicity on the SYM and on the string side is the second motivation of this paper.

2. Transformation properties of the gravity dual

The supergravity AdS/CFT dual of the β-deformed gauge theory was constructed by Lunin and Maldacena [2] by applying a solution generating SL$(3,\mathbb{R})$ transformation to the $AdS_5 \times S^5$ background, or equivalently by a STsTS^{-1} transformation. As its gauge theory dual, the supergravity solution depends on the two complex parameters, τ_0 and β, which give four real constants. In describing the Lunin-Maldacena solution as well as its string theory generalization we will denote real and imaginary parts of these parameters as

$$\tau_0 = \tau_{01} + i\tau_{02}\,, \qquad \beta = \beta_1 + i\beta_2\ . \tag{2.1}$$

[2]This parameter h in the shift on the right hand side of (1.4) is precisely the coefficient in front of the superpotential in the deformed theory. In the undeformed theory this additional shift disappears since $h = g$.

The fifth real parameter of the supergravity solution is the (quantized) radius R_E which in $\sqrt{\alpha'}$ units is

$$R_E^4 = 4\pi N \gg 1 \,. \tag{2.2}$$

We further define functions Q, g_{0E}, G and H which depend on the coordinates μ_i of the deformed sphere $\tilde{S}^5$,

$$Q := \mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2 \,, \qquad g_{0E} := R_E^4\, Q \,, \tag{2.3}$$

$$G^{-1} := 1 + \frac{|\beta|^2}{\tau_{02}} g_{0E} \,, \qquad H := 1 + \frac{{\beta_2}^2}{\tau_{02}} g_{0E} \,. \tag{2.4}$$

The Lunin-Maldacena solution, eqs. (3.24)-(3.29) of Ref [2], is written in terms of these functions. The metric in the Einstein frame is the warped product of the AdS_5 factor and the deformed 5-sphere, $\tilde{S}^5$,

$$ds_E^2 = R_E^2\, G^{-1/4} \left[ds_{AdS_5}^2 + \sum_{i=1}^{3} (d\mu_i^2 + G\mu_i^2 d\varphi_i^2) + \frac{|\beta|^2}{\tau_{02}}\, R_E^4\, G\mu_1^2\mu_2^2\mu_3^2 (\sum_{i=1}^{3} d\varphi_i)^2 \right] \,, \tag{2.5}$$

where the sphere is parameterized by the three angles φ_i and the three radial variables μ_i, which satisfy the condition $\sum_{i=1}^3 \mu_i^2 = 1$. When $\beta = 0$, the sphere is the undeformed S^5.

The dilaton, ϕ, and the axion, χ, fields of the Lunin-Maldacena solution are given by [2]

$$e^{-\phi} = \tau_{02}\, G^{-1/2}\, H^{-1} \,, \qquad \chi = \tau_{01} - \beta_1\beta_2\, H^{-1}\, g_{0E} \,. \tag{2.6}$$

In addition, the solution contains the two-form fields (which were absent in the undeformed $\beta = 0$ case)

$$B_2^{\rm NS} = \frac{\beta_1}{\tau_{02}}\, R_E^4\, G\, w_2 + 12\, \frac{\beta_2}{\tau_{02}}\, R_E^4\, w_1\, d\psi \,, \tag{2.7}$$

$$C_2 = \left(\beta_2 + \frac{\tau_{01}}{\tau_{02}}\,\beta_1\right) R_E^4\, G\, w_2 - 12 \left(\beta_1 - \frac{\tau_{01}}{\tau_{02}}\,\beta_2\right) R_E^4\, w_1\, d\psi \,, \tag{2.8}$$

as well as the usual five-form field-strength

$$F_5 = 4R_E^4\, (\omega_{AdS_5} + G\, \omega_{S^5}) \,. \tag{2.9}$$

The forms w_1, w_2, $d\psi$ and ω_{S^5} used in eqs. (2.7)–(2.9) above involve only the coordinates on the deformed sphere which can be found in [2]. These expressions will not be needed for our purposes.

It is important to stress that the dilaton and axion fields in (2.6) are not constant and should be distinguished from the corresponding field theory constant values τ_{02} and τ_{01}. This implies that the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of the IIB string theory, which acts on the dilaton-axion field τ

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \,, \qquad ad - bc = 1,\ a,b,c,d \in Z \,, \qquad \tau := \chi + ie^{-\phi} \,. \tag{2.10}$$

should be distinguished from the $\mathrm{SL}(2,\mathbb{Z})$ transformations on the gauge theory side in (1.3) which act on the parameters in (2.1). This distinction is specific to the β-deformed theory, since for $\beta = 0$ the dilaton-axion field τ was equal to the SYM coupling τ_0, and the two $\mathrm{SL}(2,\mathbb{Z})$ symmetries were one and the same.

2.1 S-duality group

We can now test how the Lunin-Maldacena solution above transforms under the SYM $\mathrm{SL}(2,\mathbb{Z})$ transformation of (1.3). One would hope that the solution would transform covariantly, and that the action of the $\mathrm{SL}(2,\mathbb{Z})$ of (1.3) on the parameters of the solution, would give the original solution transformed under the string theory $\mathrm{SL}(2,\mathbb{Z})$ of (2.10). This would imply that the metric and the 5-form are invariant, that the two 2-forms transform as a doublet, and that the τ-field transforms as in (2.10). We will show now, that this expectation holds on the Lunin-Maldacena solution *except* for the τ-field.

We first note that under the action of the SYM $\mathrm{SL}(2,\mathbb{Z})$ of (1.3), the functions Q, g_{0E} and G (but not H) in (2.3)–(2.4) are invariant:

$$Q(\tau_0',\beta') = Q(\tau_0,\beta)\,, \qquad g_{0E}(\tau_0',\beta') = g_{0E}(\tau_0,\beta)\,, \qquad G(\tau_0',\beta') = G(\tau_0,\beta)\ . \tag{2.11}$$

This implies that the metric (2.5) and the 5-form (2.9) of the Lunin-Maldacena solution are also invariant under this $\mathrm{SL}(2,\mathbb{Z})$, as expected,

$$ds_E^2(\tau_0',\beta') = ds_E^2(\tau_0,\beta)\,, \qquad F_5(\tau_0',\beta') = F_5(\tau_0,\beta)\ . \tag{2.12}$$

Furthermore, it can be shown that the two 2-forms (2.7)–(2.8) do transform as a doublet under the SYM $\mathrm{SL}(2,\mathbb{Z})$ of (1.3), again in agreement with what is expected from the action of the string theory $\mathrm{SL}(2,\mathbb{Z})$:

$$\begin{pmatrix} C_2(\tau_0',\beta') \\ B_2^{\mathrm{NS}}(\tau_0',\beta') \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2(\tau_0,\beta) \\ B_2^{\mathrm{NS}}(\tau_0,\beta) \end{pmatrix}\ . \tag{2.13}$$

To verify this transformation, it is convenient to use the fact that the following combination of the parameters transforms as a doublet under (1.3),

$$\begin{pmatrix} \beta_2 + \tau_{01}\beta_1/\tau_{02} \\ \beta_1/\tau_{02} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \beta_2 + \tau_{01}\beta_1/\tau_{02} \\ \beta_1/\tau_{02} \end{pmatrix}\,, \tag{2.14}$$

$$\begin{pmatrix} \beta_2/\tau_{02} \\ \beta_1 - \tau_{01}\beta_2/\tau_{02} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \beta_2/\tau_{02} \\ \beta_1 - \tau_{01}\beta_2/\tau_{02} \end{pmatrix}\ . \tag{2.15}$$

It is, however, impossible to make the dilaton-axion field $\tau = \chi + ie^{-\phi}$ in (2.6) transform according to eq. (2.10) by acting with (1.3) on the parameters τ_0 and β. In fact, after the passive action of (1.3), the resulting τ field does not have any easily recognizable form. This breaks the $\mathrm{SL}(2,\mathbb{Z})$ covariance of the Lunin-Maldacena supergravity solution.

In the following section we will modify the Lunin-Maldacena solution in such a way, that the passive action of the $\mathrm{SL}(2,\mathbb{Z})$ transformation (1.3) will be equivalent to the active $\mathrm{SL}(2,\mathbb{Z})$ (2.10) on the configuration itself. This reconciliation of the two $\mathrm{SL}(2,\mathbb{Z})$ transformation on the string theory background would imply then that there is a single $\mathrm{SL}(2,\mathbb{Z})$ duality symmetry which acts on the gauge theory side and on the string theory side. The action of this $\mathrm{SL}(2,\mathbb{Z})$ on the two sides of the correspondence is in terms of the different variables: τ_0 and β on the SYM side, and the supergravity fields τ and the 2-forms on the string theory side. However the symmetry is the same, and the $\mathrm{SL}(2,\mathbb{Z})$ duality of the β-deformed gauge theory implies the $\mathrm{SL}(2,\mathbb{Z})$ duality of the string theory, and vice versa.

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2.2 Periodicity properties

The second observation concerns the fact that the Lunin-Maldacena supergravity dual (2.5)–(2.9) does not automatically exhibit the desired periodicity in the deformation parameter β

$$\beta \to \beta + 1 , \qquad \beta \to \beta + \tau_0 . \tag{2.16}$$

Here we need to make two clarifying comments following [2]: first, is that being a solution of supergravity, the configuration (2.5)-(2.9) is not even expected to be periodic under $\beta \to \beta + 1$. This is because in going from β to $\beta + 1$ one would have to go outside the region of $\beta \ll 1$ where we trust the supergravity solution. The second comment is that in spite of this fact, the Lunin-Maldacena configuration for real values of β is formally invariant under a combination of the $\beta \to \beta+1$ and the $\mathrm{SL}(2,\mathbb{Z})$ T-duality transformation.

Although with the help of the T-duality $\mathrm{SL}(2,\mathbb{Z})$ of the string theory, the Lunin-Maldacena background is formally periodic under $\beta \to \beta + 1$, this periodicity is realized trivially[3] in the SYM side. Hence, one may wonder if there is a different way to implement the periodicity condition $\beta \to \beta+1$ directly in the string dual background. In the following section we will extend the configuration in (2.5)–(2.9) to make it manifestly periodic under both transformations in (2.16). The resulting β-periodic and $\mathrm{SL}(2,\mathbb{Z})$ covariant configuration of supergravity fields now faithfully represents the symmetries of the β-deformed gauge theory. In the limit of $\beta \ll 1$ it will also coincide with the Lunin-Maldacena supergravity dual.

Our construction does not attempt to give a unique solution for the β-periodic and $\mathrm{SL}(2,\mathbb{Z})$ covariant configuration. Our main point is to demonstrate that such configurations exist and to describe a procedure for constructing such configurations.

We will interpret the resulting configurations of supergravity fields as candidates for the *string* dual background to the β-deformed gauge theory. By construction our configurations are $\mathrm{SL}(2,\mathbb{Z})$ rather than $\mathrm{SL}(2,\mathbb{R})$ covariant. We conjecture that the solution of equations arising from the string theory effective action to all orders in the α' expansion and in string loops is covered by our general construction.

Finally, in section 4 we will show that our string configuration gives predictions for the string theory effective action which are in agreement with the multi-instanton calculations of appropriate correlation functions in the β-deformed gauge theory. These Yang-Mills calculations are taken from [3] and are carried out in a weakly coupled gauge theory in the large-N limit and to all orders in the deformation parameter β.

Our conclusion will be presented in section 5.

3. String configuration, $\mathrm{SL}(2,\mathbb{Z})$ covariance and periodicity in β

In this section we will modify the supergravity dual of eqs. (2.5)–(2.9) to satisfy

- the double periodicity in β of (2.16),

[3] In particular, the weak-coupling instanton expression for the τ field in (4.10) is automatically periodic and does not require any additional $\mathrm{SL}(2,\mathbb{Z})$ transformation on the S^5 sphere generated by instanton collective coordinates χ_{AB}.

- the $SL(2,\mathbb{Z})$ covariance dictated by eq. (2.10) when the parameters of the configuration are transformed according to eq. (1.3). Furthermore, we request that
- the configuration reduces to the supergravity solution of Lunin and Maldacena (2.5)–(2.9) in the small β limit $|\beta| \ll 1$ and
- it agrees with the Yang-Mills instanton prediction in the limit of weak coupling $\tau_{02} \to \infty$ and large N.

3.1 Metric and form-fields

The modification needed for the metric (2.5), two-forms (2.7)–(2.8), and the five-form fields (2.9) is relatively straightforward. We have shown in the previous section that the Lunin-Maldacena expressions for these fields already transform correctly under the $SL(2,\mathbb{Z})$. We need to keep this property and ensure that the fields satisfy the required periodicity in β. This can be achieved by replacing β in the original expressions (2.4), (2.5), (2.7)–(2.9) by a function $\mathcal{B}(\beta,\bar\beta,\tau_0,\bar\tau_0)$

$$\beta \longrightarrow \mathcal{B}(\beta,\bar\beta,\tau_0,\bar\tau_0)\ . \tag{3.1}$$

The function $\mathcal{B}$ will be chosen to be doubly periodic in β, in agreement with (2.16), and to transform under the $SL(2,\mathbb{Z})$ of (1.3) as

$$\mathcal{B}\left(\frac{\beta}{c\tau_0+d},\frac{\bar\beta}{c\bar\tau_0+d},\frac{a\tau_0+b}{c\tau_0+d},\frac{a\bar\tau_0+b}{c\bar\tau_0+d}\right)=\frac{\mathcal{B}(\beta,\bar\beta,\tau_0,\bar\tau_0)}{c\tau_0+d}\ . \tag{3.2}$$

We also require that

$$\mathcal{B}\ \to\ \beta, \quad \text{for } |\beta| \ll 1\ ,\ \tau_0 \text{ arbitrary}, \tag{3.3}$$

so that the Lunin-Maldacena solution is recovered in the small β limit. The above properties do not fix $\mathcal{B}$ uniquely. To further constrain $\mathcal{B}$, we need a more detailed knowledge of the string dynamics. As already mentioned earlier, In this paper, we will be satisfied instead with demonstrating that it is possible to construct such a solution.

The simplest solution can be constructed by assuming that $\mathcal{B}$ depends on β and τ_0 analytically, so that

$$\mathcal{B}\left(\frac{\beta}{c\tau_0+d},\frac{a\tau_0+b}{c\tau_0+d}\right)=\frac{\mathcal{B}(\beta,\tau_0)}{c\tau_0+d}\ . \tag{3.4}$$

Thus, in this case we will construct the string configuration using elliptic functions. As we will show below, we can give a concise specification of $\mathcal{B}$ in this case. If, on the other hand, the function $\mathcal{B}$ is not analytic, then our approach outlined in (3.1)–(3.3) still holds, we just cannot fully specify the form of $\mathcal{B}$. We will outline below how to construct the analytic function $\mathcal{B}(\beta,\tau_0)$ with the desired properties. The reader not interested in this discussion can skip directly to eq. (3.19).

Elliptic functions have the required double periodicity (2.16) and are meromorphic functions. For a review of elliptic functions one can consult for example ref. [25]. It is known that the number of poles of an elliptic function in a period parallelogram, counting multiplicity, cannot be less than two. The simplest elliptic functions thus either have one

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pole of second order, or two distinct poles of first order. Weierstrass function is of the first type, while the Jacobi elliptic functions, sn, cn, dn, are of the second type. In the following we will construct our candidate for the modified string background using the Weierstrass function.[4] We recall that the Weierstrass function is a function of two complex variables and is defined by

$$\mathscr{P}(\beta, \tau_0) := \frac{1}{\beta^2} + \sum_{\lambda \in \Lambda/\{0\}} \left(\frac{1}{(\beta - \lambda)^2} - \frac{1}{\lambda^2} \right), \tag{3.5}$$

Here the lattice $\Lambda = m + n\tau_0$ is generated by 1 and τ_0. The variable τ_0 is defined on the upper half plane. By definition (3.5), the Weierstrass function satisfies the required periodicity in the variable β,

$$\mathscr{P}(\beta + 1, \tau_0) = \mathscr{P}(\beta + \tau_0, \tau_0) = \mathscr{P}(\beta, \tau_0) \ . \tag{3.6}$$

Furthermore, under the $\mathrm{SL}(2, \mathbb{Z})$ transformation (1.3) of τ_0 and β, $\mathscr{P}$ transforms as a modular function of weight 2

$$\mathscr{P}\left(\frac{\beta}{c\tau_0 + d}, \frac{a\tau_0 + b}{c\tau_0 + d} \right) = (c\tau_0 + d)^2 \, \mathscr{P}(\beta, \tau_0) \ . \tag{3.7}$$

One can also define the derivative of the Weierstrass function with respect to the first argument. The derivative is a modular function of weight 3, that is

$$\mathscr{P}'\left(\frac{\beta}{c\tau_0 + d}, \frac{a\tau_0 + b}{c\tau_0 + d} \right) = (c\tau_0 + d)^3 \, \mathscr{P}'(\beta, \tau_0) \ . \tag{3.8}$$

For completeness, we note that $\mathscr{P}(\beta)$ satisfies the differential equation

$$\mathscr{P}(\beta)'^2 = 4\mathscr{P}^3(\beta) - g_2 \mathscr{P}(\beta) - g_3 \, , \tag{3.9}$$

where the coefficients depend only on τ_0,

$$g_2 = 60 E_4(\tau_0) \, , \quad g_3 = 140 E_6(\tau_0) \ . \tag{3.10}$$

Here $E_{2k}(\tau_0)$ denotes the Eisenstein series

$$E_{2k}(\tau_0) = \sum_{\lambda \in \Lambda/\{0\}} \frac{1}{\lambda^{2k}} \, , \quad \text{for positive integer } k \geq 2 \ . \tag{3.11}$$

E_{2k} is a modular form[5] of weight $2k$

$$E_{2k}\left(\frac{a\tau_0 + b}{c\tau_0 + d} \right) = (c\tau_0 + d)^{2k} \, E_{2k}(\tau_0) \, , \tag{3.12}$$

[4]Note that, unlike the Weierstrass function, the Jacobi elliptic functions do not have simple modular transformation like (3.7) below.

[5]A modular form is holomorphic on the upper half plane. Modular function is needed to be meromorphic only.

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and has the (weak-coupling) expansion near $q = e^{i\pi\tau_0} \sim 0$,

$$E_{2k} = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{m=1}^{\infty} \sigma_{2k-1}(m) q^m \,, \tag{3.13}$$

where $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$ for Re$s > 1$ is the Riemann zeta function, and $\sigma_n(m) = \sum_{d|m} d^n$.

We can now construct the function $\mathcal{B}$ in eq. (3.4) in terms of elliptic functions. The general theory of elliptic function states that every elliptic function can be expressed in terms of the $\mathscr{P}$ and its derivative $\mathscr{P}'$ in the form [25]

$$f(\beta) = R_1(\mathscr{P}) + R_2(\mathscr{P})\mathscr{P}', \tag{3.14}$$

where R_1 and R_2 are rational functions of their arguments. From the transformation properties (3.7), (3.8) and (3.4), one can easily conclude that B has to be of the form

$$\mathcal{B} = \frac{a_1 \, \mathscr{P}'}{\mathscr{P}^2 + E} \,, \tag{3.15}$$

where $E = E(\tau_0)$ is a modular function of weight 4 and $a_1 = a_1(\tau_0)$ is a modular invariant function. The formula above has to be consistent with the small β behaviour (3.3). The asymptotic expansion of $\mathscr{P}$ for small β is given by

$$\mathscr{P}(\beta, \tau_0) = \beta^{-2} + \frac{g_2}{20} \beta^2 + \mathcal{O}(\beta^4) \quad \text{for } |\beta| \ll 1 \,. \tag{3.16}$$

Thus (3.3) is reproduced if we set

$$a_1 = -1/2 \,. \tag{3.17}$$

As for E, it is natural to require that it is analytic in τ_0, so that E is a modular form[6] of weight 4. We note that the set $\mathcal{M}_{2k}$ of all modular forms of a given weight $2k$ is a finite dimensional linear space over the complex field. The dimension of M_{2k} is given by

$$\dim \mathcal{M}_{2k} = \begin{cases} [k/6] \,, & \text{if } k \equiv 1 \pmod 6 \\ [k/6] + 1, & \text{otherwise} \,. \end{cases} \tag{3.18}$$

In particular, $\mathcal{M}_4 = \mathbb{C} E_4$. Thus $\mathcal{B}$ in eq. (3.1) is given by

$$\mathcal{B}(\beta, \tau_0) = -\frac{1}{2} \frac{\mathscr{P}'(\beta, \tau_0)}{\mathscr{P}^2(\beta, \tau_0) + \text{const} \cdot E_4(\tau_0)} \,, \tag{3.19}$$

where const is some constant.

[6] In general $E(\tau_0)$ can also depend on the space-time coordinates, or at least on the coordinates $\mathbf{x}$ of the deformed sphere. For simplicity, we will assume that there is no such dependence, and $E(\tau_0)$ is a coordinate-independent modular form. In a sense, we are building up the simplest generalization of the Lunin-Maldacena solution which satisfies all the desired criteria. More general configurations are always possible.

It is always possible to construct more general solution $\mathcal{B}$ without assuming analyticity. However as shown in [3] and reviewed in section 4 below, the instanton result (4.10) predicts that the string dilaton-axion field τ is analytic in β (at least in the leading order at weak coupling). It is less clear why the the metric and the form-fields in the string dual background should be constructed using a holomorphic "renormalization" of β to $\mathcal{B}(\beta, \tau_0)$ as in our simple example above. In general, one can follow the more general procedure in eqs. (3.1)–(3.3) which involves non-analytic functions $\mathcal{B}(\beta, \bar{\beta}, \tau_0, \bar{\tau}_0)$. A better understanding of the string dynamics is then needed to fully constrain the choice of $\mathcal{B}(\beta, \bar{\beta}, \tau_0, \bar{\tau}_0)$.

In section 3.2 we will construct the dilaton-axion field τ. Before closing this subsection, we would like to make a few general comments.

First we want to comment on the origin of the modifications of the supergravity dual. We recall that the Lunin-Maldacena solution with real β, can be obtained by the TsT transformation[7] [2] from the undeformed $AdS_5 \times S^5$ solution. Although $AdS_5 \times S^5$ is an exact background of string theory, the generated solution does not admit the required double periodicity[8] and S-duality of the field theory. Hence, the supergravity dual cannot be an exact string theory background even for real-valued deformations β. In fact, it is known that T-duality transformations for curved backgrounds are generally modified by α' and string loop effects. Therefore the modifications of the string solution from the original Lunin-Maldacena solution can be attributed to the corrections to the form of the T-duality transformation. We note that these corrections are becoming more significant as β increases and this is the behaviour to be expected as large β corresponds to large curvature corrections (α').

It has been argued in [10, 11] that for rational β, the gauge theory is dual to an orbifold with a discrete torsion.[9] Furthermore, Lunin and Maldacena have demonstrated [2] that their supergravity solution with rational values of the β parameter is also related to the orbifold description via an action of the $\mathrm{SL}(2, \mathbb{Z})$ T-duality. Our solution agrees with that of Lunin-Maldacena for small β and hence its connection with the orbifold is guaranteed in this limit. However for generic rational β, it is not clear how to relate our solution to the orbifold description because the exact form of the T-duality rules is not available. In principle, the requirement of matching to the orbifold description at rational values of β should give and interesting constraint on the form of the T-duality rule. It is possible that one may be able to derive the exact T-duality rules in this way.

Finally, in spite of the fact that for rational values of β the dual geometry involves the orbifold, we expect that the geometrical description in terms of the $AdS_5 \times \tilde{S}^5$ should hold throughout the parameter space, including these rational values of β. In gauge theory in the conformal phase (which is what is relevant to our AdS/CFT correspondence) we do not expect to see any discontinuities in the results as β becomes rational.[10]

[7] Where T is a T-duality and s is a shift.

[8] In particular, the dilaton-axion field τ in the Lunin-Maldacena background does not agree with the SYM instanton expression (4.10) unless β is small.

[9] To be precise, $Z_n \times Z_n$ orbifold when $\beta = m/n$.

[10] All known instanton-generated as well as perturbative results in the SYM in the conformal phase depend on β smoothly. The story is different on the Coulomb branch where the F-term constraints resemble the

3.2 Dilaton-axion pair

Next we consider the string dilaton-axion field $\tau = \chi + ie^{-\phi}$. In section 2.1 we found that the Lunin-Maldacena expression for τ in eq. (2.6) did not transform covariantly under the $\mathrm{SL}(2,\mathbb{Z})$. We want to construct the τ field which does. Furthermore, the resulting τ should satisfy the weak-coupling and the small-β limits. Writing τ as

$$\tau = \tau_0 + \delta \; . \tag{3.20}$$

The fact that τ should agree with the original Lunin-Maldacena solution in the small β limit (for arbitrary τ_0) gives:

$$\delta = \frac{i\beta^2}{2}\, g_{0E}\,, \qquad\qquad \text{for } |\beta| \ll 1 \text{ and } \tau_0 \text{ arbitrary.} \tag{3.21}$$

This equation is the leading order term in β in the small β expansion of the Lunin-Maldacena τ field (2.6). In addition to (3.21) the modified configuration should agree with the instanton result in the weak coupling limit $\tau_{02} \to \infty$ (for arbitrary *complex* values of β):

$$\delta = \frac{ig_{0E}}{2\pi^2}\, \sin^2 \pi\beta\,, \qquad\qquad \text{for } \tau_{02} \to \infty \text{ and } \beta \text{ arbitrary.} \tag{3.22}$$

Equation (3.22) is the instanton prediction. We will discuss its origin in the following section. For now we only need to note that it is derived at weak-coupling and in the large N limit. What is important is that (3.22) is valid for arbitrary complex values of the deformation parameter β.

We note that τ is analytic in the parameters τ_0 and β in these limits. We conjecture that the string field τ is analytic in the parameters β and τ_0 in general. Therefore we will take $\delta = \delta(\beta, \tau_0, \mathbf{x})$ a function of the parameters β, τ_0, as well as of the coordinates $\mathbf{x}$ on the deformed sphere. Writing

$$\delta = \frac{u}{v}\,, \tag{3.23}$$

it is easy to show that if u, v transform under (1.3) as

$$u \to \frac{u}{(c\tau_0 + d)^2}\,, \qquad v \to v + \frac{cu}{c\tau_0 + d}\,, \tag{3.24}$$

then τ transforms as required by (2.10)

$$\tau \to \frac{a\tau + b}{c\tau + d}\; . \tag{3.25}$$

Furthermore, u and v have to be constructed in such a way that the resulting τ satisfies the weak-coupling and the small-β limits.

The transformation property of v reminds us of that of the elliptic theta function

$$\theta_1(z,\tau_0) \;:=\; 2q^{1/4} \sin \pi z \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2\pi z + q^{4n})\,, \quad q = e^{i\pi\tau_0}\; . \tag{3.26}$$

commutation relation of a noncommutative torus and admit new solutions when β is rational.

We recall that

$$\frac{\theta_1(z,\tau_0)}{\theta_1'(0,\tau_0)} \to \frac{\exp(i\pi c z^2/(c\tau_0+d))}{c\tau_0+d}\,\frac{\theta_1(z,\tau_0)}{\theta_1'(0,\tau_0)}\,, \tag{3.27}$$

which allows us to write a general solution to (3.24) as follows:

$$u = z^2\,, \qquad v = h + \frac{1}{i\pi}\ln\left(\frac{\theta_1(z,\tau_0)}{z\,\theta_1'(0,\tau_0)}\right)\,. \tag{3.28}$$

Here $z = z(\beta,\tau_0,\mathbf{x})$ is required to transform as[11]

$$z \to \frac{z}{c\tau_0+d}\,, \tag{3.29}$$

and $h(\beta,\tau_0,\mathbf{x})$ is a modular invariant function. We also choose the branch of the logarithm where $\ln 1 = 0$.

As in the above, one may represent the elliptic function u as a rational function of $\mathscr{P}$ and $\mathscr{P}'$. However only a definite combination of them can be written as a complete square z^2 with the required weight. This gives a form similar to (3.19) for $\mathcal{B}$. It is natural to assume that z is simply proportional to $\mathcal{B}$

$$z = b_1\,\mathcal{B}\,. \tag{3.30}$$

where $b_1(\tau_0)$ is a modular invariant function.[12] z is doubly periodic in β.

Now we examine the limits. In the leading order at small β, we find

$$z = \beta\, b_1\,, \quad v = h(0,\tau_0,\mathbf{x})\,. \tag{3.31}$$

Thus the matching with the Lunin-Maldacena expression for τ in the small β limit gives

$$h(0,\tau_0,\mathbf{x}) = -2ib_1(\tau_0)^2/g_{0E}\,. \tag{3.32}$$

Next we look at the weak coupling limit. We have

$$\mathscr{P}(\beta,\tau_0) = \pi^2\left(\frac{1}{\sin^2\pi\beta} - \frac{1}{3}\right), \qquad E_4(\tau_0) = \frac{\pi^4}{45}\,, \qquad \tau_{02}\to\infty\,, \tag{3.33}$$

and thus $\mathcal{B}$ is independent of τ_0 in the limit. Therefore if we choose b_1 such that it becomes infinite in the weak-coupling limit, and also set h in (3.28) to be

$$h = z^2 f \tag{3.34}$$

for some modular function f of weight 2 which remains finite in the limit, then the log term in v in (3.28) can be neglected. Hence $u/v = 1/f$ in this limit. This can be achieved

[11]We have chosen to define u in terms of z, rather than use $\sqrt{u}$ in the argument of the θ-function in the second equation in (3.28). This is because the function $\sqrt{u}$ is multi-valued. A branch for the square root has to be chosen in order to define v in (3.28). However, the branch cut is not invariant under (3.29). Therefore we conclude that (3.28) is well-defined only if u is a complete square and hence z is single-valued.

[12]Again, on general grounds, it is possible that $b_1(\tau_0)$ also depends on the space-time coordinates $\mathbf{x}$. In what follows we will assume a simple scenario where b_1 depends only on τ_0.

by taking b_1 to be, for example, the modular function $J(\tau_0) := g_2^2/(g_2^3 - 27g_3^2)$ since it has a double pole at $q = 0$

$$J(\tau_0) = (1728)^{-1} q^{-2} + \text{finite} \; . \tag{3.35}$$

To match with the instanton result (3.22), the function f has to satisfy

$$\lim_{\tau_{02}\to\infty} f(\beta, \tau_0, \mathbf{x}) = \frac{-2i\pi^2}{g_{0E} \sin^2 \pi\beta} \; . \tag{3.36}$$

This can be satisfied by

$$f(\beta, \tau_0, \mathbf{x}) = \frac{-2i}{g_{0E}} (\mathscr{P} + G(\tau_0)) \, , \tag{3.37}$$

where $G(\tau_0)$ is a modular function of weight 2 such that $\lim_{\tau_{02}\to\infty} G(\tau_0) = \pi^2/3$. This can be constructed from the Eisenstein series, for example, $G(\tau_0) = \frac{7E_6}{2E_4}$. One can check easily that (3.32) is also satisfied.

Summarizing, we have constrained the modified string configuration using the requirement of double periodicity in β, the SL$(2, \mathbb{Z})$ symmetry, and matching to the known asymptotic form of the supergravity solution in the small β and in the weak coupling limits. The result is not uniquely fixed without further input of the full string dynamics. A particularly simple solution can be written down with the assumptions of analyticity. The result is that for the metric and the 2-form fields, one has to replace the β parameter in the original Lunin-Maldacena configuration by the function $\mathcal{B}$ of (3.19)

$$\mathcal{B}(\beta, \tau_0) = -\frac{1}{2} \frac{\mathscr{P}'(\beta, \tau_0)}{\mathscr{P}^2(\beta, \tau_0) + \text{const} \cdot E_4(\tau_0)} \; . \tag{3.38}$$

For the dilaton-axion field we have the following expression,

$$\tau(\beta, \tau_0, \mathbf{x}) = \tau_0 \; - \; i \left[\frac{-2}{g_{0E}(\mathbf{x})} \Big(\mathscr{P}(\beta, \tau_0) + G(\tau_0) \Big) + \frac{1}{\pi z^2} \ln \left(\frac{\theta_1(z, \tau_0)}{z \; \theta_1'(0, \tau_0)} \right) \right]^{-1} , \tag{3.39}$$

where

$$z = b_1(\tau_0) \, \mathcal{B}(\beta, \tau_0) \; . \tag{3.40}$$

Here b_1 is a modular invariant function which blows up in the weak coupling limit $\tau_{02} \to \infty$. A simple choice for b_1 would be $b_1 = J(\tau_0)$. Finally, $G(\tau_0)$ is a modular function of weight 2.

We remark that in establishing the classical integrability of the Lunin-Maldacena background (for real β) in [18], the knowledge of the dilaton-axion pair was never used, and apart from the dilaton-axion pair, our proposed string background is the same as Lunin-Maldacena with the substitution $\beta \to \mathcal{B}$. Therefore the classical integrability (for real β) apply equally for any $\mathcal{B}(\beta, \bar{\beta}, \tau_0, \bar{\tau}_0)$ with the property that $\mathcal{B}$ is real whenever β is real.

4. String theory effective action and instantons

The string effective action $S_{\rm IIB}$ is related via the AdS/CFT holographic formula [20, 21] to correlation functions in the gauge theory,

$$\exp -S_{\rm IIB} \left[\Phi_{\mathcal{O}}; J\right] = \left\langle \exp \int d^4x \, J(x) \mathcal{O}(x) \right\rangle \; . \tag{4.1}$$

Here $\Phi_{\mathcal{O}}$ are Kaluza-Klein modes of the supergravity fields which are dual to composite gauge theory operators $\mathcal{O}$. The boundary conditions of the supergravity fields are set by the gauge theory sources on the boundary of AdS_5.

The effective action $S_{\rm IIB}$ of type IIB string theory is invariant under the $\mathrm{SL}(2,\mathbb{Z})$ transformations (2.10). The action of this symmetry leaves the metric invariant, but acts upon the dilaton-axion field τ, as well as on the 2-form fields. It is well-known that in the supergravity approximation, the action is invariant under the full $\mathrm{SL}(2,\mathbb{R})$ symmetry, however the higher-derivative corrections to it, as well as the string-loop corrections break the $\mathrm{SL}(2,\mathbb{R})$ symmetry down to $\mathrm{SL}(2,\mathbb{Z})$.

At leading order beyond the Einstein-Hilbert term in the derivative expansion, the IIB effective action is expected to contain [22, 23] the $\mathcal{R}^4$ term

$$(\alpha')^{-1} \int d^{10}x \sqrt{-g_{10}}\, e^{-\phi/2}\, f_4(\tau,\bar\tau)\, \mathcal{R}^4 \,, \tag{4.2}$$

as well as its many superpartners, including a totally antisymmetric 16-dilatino effective vertex of the form

$$(\alpha')^{-1} \int d^{10}x \sqrt{-g_{10}}\, e^{-\phi/2}\, f_{16}(\tau,\bar\tau)\, \Lambda^{16} \ + \ \text{H.c.} \tag{4.3}$$

D-instanton contributions in supergravity contribute precisely to eqs. (4.2)–(4.3) and their superpartners, i.e. to the leading order higher-derivative corrections to the classical IIB supergravity [23]. The D-instanton contribution to an n-point interaction of supergravity fields comes from a tree level Feynman diagram with one vertex located at a point $(x_0, \rho, \hat\Omega)$ in the bulk of $AdS_5 \times \tilde S^5$. The diagram also has n external legs connecting the vertex to operator insertions on the boundary. We will outline below how to single out these D-instanton contributions in (4.2)–(4.3).

We further note that the higher-derivative corrections eqs. (4.2)–(4.3) must respect the $\mathrm{SL}(2,\mathbb{Z})$ of eq. (2.10). Under these transformations supergravity field components Φ acquire (discrete) phases,

$$\Phi \ \longrightarrow \ \left(\frac{c\tau+d}{c\bar\tau+d}\right)^{-\,q_\Phi/2} \Phi \,, \tag{4.4}$$

The charge q_Φ for the dilatino is $3/2$ and for the $\mathcal{R}$ field it is zero.

Equations (4.2)–(4.3) are written in the string frame with the coefficients $f_n(\tau,\bar\tau)$ being the modular forms of weights $((n-4), -(n-4))$ under the $\mathrm{SL}(2,\mathbb{Z})$ transformations (2.10),

$$f_n(\tau,\bar\tau) \ := \ f^{\,(n-4),-(n-4)}(\tau,\bar\tau) \ \longrightarrow \ \left(\frac{c\tau+d}{c\bar\tau+d}\right)^{n-4} f^{\,(n-4),-(n-4)}(\tau,\bar\tau)\,. \tag{4.5}$$

The modular properties of f_n precisely cancel the phases of fields in (4.4) acquired under the $\mathrm{SL}(2,\mathbb{Z})$. Thus the full string effective action is invariant under the $\mathrm{SL}(2,\mathbb{Z})$ and this modular symmetry ensures the S-duality of the type IIB superstring.

The modular forms f_n have been constructed by Green and Gutperle in [22]. In the weak coupling expansion the expressions for f_n contain an infinite sum of exponential terms

$$e^{-\phi/2}\, f_n \ \ni \ \sum_{k=1}^{\infty} \text{const}\cdot \left(\frac{k}{G^{1/2}\, g^2}\right)^{n-7/2} e^{2\pi i k\tau} \sum_{d|k} \frac{1}{d^2}\,, \tag{4.6}$$

It is clear that in the sum above each term corresponds to a contribution of a D-instanton of charge k. On the other hand, k-instanton contributions can also be calculated directly in gauge theory. It was shown recently in ref. [3] that each of the terms in the sum in the expression above can be identified with a contribution of an instanton of charge k in the β-deformed SYM theory.[13] This precise identification was performed in [3] at weak coupling in the large-number of colours $N \gg 1$ and the small-deformation $\beta \ll 1$ limits. The latter limit in particular, was necessary to ensure that the Lunin-Maldacena supergravity dual remains a valid approximation to string theory [2]. Here we will generalize these results to arbitrary values of the deformation parameter β.

We now consider the exponential factors $e^{2\pi i k\tau}$ in the eq. (4.6), and compare them with the relevant (generic) part of the multi-instanton contributions to correlators of the β-deformed gauge theory calculated in [3]. The generic Yang-Mills k-instanton contributions have been recently calculated in [3] and are of the form:

$$\mathcal{F}_k := e^{-k\frac{8\pi^2}{g^2}+ik\theta} \left(1 - 4\sin^2(\pi\beta)\,Q\right)^{k(N-2)} . \tag{4.7}$$

This expression was derived [3] in the weak-coupling limit in gauge theory and is valid for arbitrary values of the complex deformation parameter β. Here Q is the same function of the μ_i coordinates on the deformed sphere $\tilde{S}^5$ as the one appearing in (2.3). These coordinates and the sphere $\tilde{S}^5$ itself arise in the Yang-Mills instanton approach from the bosonic collective coordinates χ_{AB} of the instanton, which are used to bi-linearize the term in the instanton effective action of the fourth order in fermionic collective coordinates. We refer the reader to sections 4 and 8 of ref. [3] (or section 4 of ref. [6]) for more detail.

This k-instanton factor $\mathcal{F}_k$ is supposed to match with the k D-instanton term $e^{2\pi i k\tau}$ in (4.6). We thus have,

$$\tau = \tau_0 + \frac{N}{2\pi i}\log\left(1 - 4\,Q\,\sin^2\pi\beta\right) . \tag{4.8}$$

We can now simplify the instanton prediction above if we recall that the instanton measure includes the integration over all instanton collective coordinates, including the integration over the sphere $\tilde{S}^5$. These integrations are supposed to be carried out in the limit of large number of colours, $N \to \infty$. The function Q on the right hand side of (4.8) is a function of the collective coordinates, which themselves are integration variables. Schematically, we have

$$\int d\mu_1\, d\mu_2\, d\mu_3\, \delta(\mu_1^2+\mu_2^2+\mu_3^2)\, e^{2\pi i k\tau}$$
$$= \int d\mu_1\, d\mu_2\, d\mu_3 \delta(\mu_1^2+\mu_2^2+\mu_3^2) \exp\left[2\pi i k\tau_0 + kN\log\left(1-4Q\sin^2\pi\beta\right)\right] . \tag{4.9}$$

[13] The result of [3] is a generalization to the β-deformed case of earlier instanton calculations carried out in the undeformed $\mathcal{N}=4$ SYM. These calculations have been performed in [4, 5] at the 1-instanton level and in [24, 6] for the general k-instanton case. Remarkably, these gauge theory results, including the results of [3] in the β-deformed theory are in precise agreement with the supergravity predictions for the effective action eqs. (4.2)–(4.6). In what follows it will be sufficient to concentrate only on reproducing the exponential factors $e^{2\pi i k\tau}$ on the right hand side of eq. (4.6).

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In the $N \to \infty$ limit, the integral can be performed in the saddle point approximation. This selects the dominant value of the function $Q(\mu_i)$ to be $\langle Q \rangle \sim 1/N \ll 1$. This implies that the logarithm in (4.8) can be power-expanded and only the leading order term in Q should be kept in anticipation of the integrations over collective coordinates. This gives the effective instanton prediction for τ field in string theory in the form:

$$\tau = \tau_0 + i\,\frac{g_{0E}}{2\pi^2}\,\sin^2\pi\beta\,, \tag{4.10}$$

where we have used $g_{0E} = 4\pi NQ$ as in (2.3). As expected, the dependence on β is periodic as $\beta \to \beta + 1$. The second type of β-periodicity, $\beta \to \beta + \tau_0$, clearly cannot be seen in this semiclassical instanton result. Indeed, at week coupling $\tau_{02} \to \infty$, and the second periodicity is lost.

Equation (4.10) is the instanton prediction which we have used in (3.22) in constraining the form of the string configuration in the previous section.

5. Conclusions

In this paper we have constructed the generalization of the Lunin-Maldacena supergravity solution dual to the β-deformed conformal gauge theory. Our modified configuration satisfies the criteria outlined in the beginning of section 3. In particular, this configuration has a double periodicity in the deformation parameter β. It also transforms covariantly under the $\mathrm{SL}(2,\mathbb{Z})$ duality of string theory when the parameters τ_0 and β are transformed under the gauge-theory $\mathrm{SL}(2,\mathbb{Z})$. This reconciles the $\mathrm{SL}(2,\mathbb{Z})$ Montonen-Olive duality of the β-deformed SYM with the string theory $\mathrm{SL}(2,\mathbb{Z})$ invariance.

In the β-deformed case, the supergravity background receives corrections in string theory and thus should not identified with the exact string background. Since our configuration is fully consistent with the symmetries of the theory and since it transforms under the $\mathrm{SL}(2,\mathbb{Z})$ rather than the $\mathrm{SL}(2,\mathbb{R})$ symmetry, we expect it to represent the string theory (and not the supergravity) background. We propose that in comparing to the SYM side one should use this string background configuration in the string theory effective action.

We have encountered a certain degree of freedom in our construction. This freedom cannot be fixed from the symmetry requirements alone, nor by the matching with the known limits in supergravity and gauge theory. We have presented the simplest form of the solution. More general solutions can be easily constructed, and they should be tested when better knowledge of the full dual string theory dynamics is available.

Acknowledgments

We acknowledge useful discussions with Adi Armoni, Ed Corrigan, Tim Hollowood, George Georgiou, Asad Naqvi and Carlos Núñez at the outset of this work. We also thank Oleg Lunin and Juan Maldacena for comments on this paper. The research of C.S.C. is supported by EPSRC through an Advanced Fellowship. V.V.K. is supported by a PPARC Senior Fellowship.

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